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Differential Forms in Synthetic Differential Geometry

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In this paper we establish the equivalence among various kinds of pointwise forms under the presence of appropriate symmetric connections. We show also, in the presence of a diffusion on M, a bijection between pointwise n-forms and certain global n-forms called localizable.

1. INTRODUCTION

The theory of differential forms in synthetic differential geometry can be traced back to Kock (1978). It was discussed also by G. E. Reyes and was presented in books by Kock (1981) and Lavendhomme (1987).

For basic concepts of synthetic differential geometry and its notations in particular, the reader is referred to Lavendhomme (1996). The latter work distinguishes global and pointwise differential forms. A global n-form on a microlinear space M assigns a real-valued function on M to each n-tuple of vector fields on M in an n-linear and alternating way. Pointwise differential forms are divided into classical and singular ones. A classical n-form on M assigns a real number to each n-tuple of vectors tangent at the same point of M in an n-linear and alternating way, while a singular n-form on M assigns a real number to each microcube on M in an n-linear and alternating way. Lavendhomme (1996) shows that singular differential forms naturally give rise to classical ones (Section 4.1, Proposition 5), which give, in turn, global ones (Section 6.1, Proposition 2). It is also shown that if M is endowed with a symmetric connection, then there is a bijection between classical 2-forms and singular ones (Section 4.1, Proposition 6).

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Nishimura (1997) proposed a more comprehensive theory of pointwise differential forms, in which they operate on M^E not only with $E = D(n)$ (in case of classical n-forms) and $E = D^n$ (in case of singular n-forms), but also with various small objects E between the above two extremes (Section 4). However, his exposition is erroneous in Proposition 4.7 as well as in Corollary 4.8, which was pointed out by the first author. Thus we decided to take up the matter again in collaboration.

In this paper we will establish the equivalence among various kinds of pointwise forms under the presence of appropiate symmetric connections. We will show also, under the presence of a diffusion on M, a bijection between pointwise n-forms and certain global n-forms called localizable.

2. SOME VECTOR BUNDLES

For n a natural number and any $p \geq 2$, consider

$$D(n, p) = \{\underline{d} \in D^n | \forall\ i_1 \cdots i_p\ d_{i_1} \cdots d_{i_p} = 0\}$$

For $p = 2$ we have $D(n; 2) = D(n)$ and for $p > n$, $D(n; p) = D^n$. We have the inclusions i_p: $D(n; p) \to D(n; p + 1)$

Let M be a microlinear object. Each $M^{D(n;p)}$ has n scalar multiplications. They are defined by

$$(\alpha \cdot_k \gamma)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, \alpha d_k, \ldots, d_n)$$

for $1 \leq k \leq n$ and for any α in R and γ in $M^{D(n;p)}$.

We will prove that we can define additions along an axis. It is a little less simple than in the extreme cases of $M^{D(n)}$ and M^{D^n} where the definitions of the different additions are easy. In the case of $D^{D(n)}$ it is because $M^{D(n)} \simeq M^D \times_M M^D \times_M \cdots \times_M M^D$ and in the case of M^{D^n} it is because an n-microcube γ: $D^n \to M$ can be seen (in n different manners) as a tangent vector from D to $M^{D^{n-1}}$.

In the general case of $M^{D(n;p)}$, it does not seem possible to write $D(n; p)$ as a product $D \times E$ and thus γ: $D(n; p) \to M$ is not directly a tangent vector to some M. But we will see that $D(n; p)$ is a quasicolimit of a diagram composed of products of D. More precisely, consider for $1 \leq j_1 < j_2 < \cdots < j_{p-1} \leq n$ products

$$D_{j_1} \times D_{j_2} \times \cdots \times D_{j_{p-1}}$$

where $D_{j_1} = D_{j_2} = \cdots = D_{j_{p-1}} = D$. If $1 \leq i_1 < i_2 < \cdots < i_{p-2} \leq n$ and $\{i_1, \ldots, i_{p-2}\} \subseteq \{j_1, \ldots, j_{p-1}\}$. We consider also the injections

$$D_{i_1} \times \cdots \times D_{i_{p-2}} \ (= D^{p-2}) \to D_{j_1} \times \cdots \times D_{j_{p-1}} \ (= D^{p-1})$$

obtained by introducting 0 at the kth position, where k is the only natural in $\{j_1, \ldots, j_{p-1}\}$ and not in $\{i_1, \ldots, i_{p-2}\}$. We have a right cone to $D(n; p)$ thanks to the injections of $D_{j_1} \times \cdots \times D_{j_{p-1}}$ into $D(n; p)$.

For example, in the case of $D(4; 3)$ this cone is given by Fig. 1, and in the case of $D(4; 4)$, by Fig. 2.

Proposition 1. $D(n; p)$ is a quasicolimit of the diagram of Fig. 2

Proof. It is enough to prove that R believes that it is a colimit. We consider thus the (C_p^n) $(p - 1)$-microcubes:

$$\gamma_{j_1 \cdots j_{p-1}}: \quad D^{p-1} \to R$$

for $1 \leq j_1 < j_2 < \cdots < j_{p-1} \leq n$ such that the diagram commutes.

We prove that there exists a factorization by $D(n; p)$. Any $\gamma_{j_1 \cdots j_{p-1}}$ is a polynomial of degree $p - 1$ in the $d_{j_1}, \ldots, d_{j_{p-1}}$. Let $b_{j_1 \cdots j_{p-1}} d_{j_1} \cdots d_{j_{p-1}}$ its unique term of degree $p - 1$.

The commutation of the diagram gives us a coherence for the terms of degree $\leq p - 2$. For example, if $a^{i_1 \cdots i_{p-2}}_{(j_1 \cdots j_{p-1})} d_{i_1} \cdots d_{i_{p-2}}$ and $a^{i_1 \cdots i_{p-2}}_{(j'_1 \cdots j'_{p-1})} d_{j_1} \cdots d_{i_{p-2}}$ are corresponding terms of degree $p - 2$ in, respectively, $\gamma_{j_1 \cdots j_{p-1}}$ and $\gamma_{j'_1 \cdots j'_{p-1}}$, we have $\{i_1, \ldots, i_{p-2}\} \subseteq \{j_1, \ldots, j_{p-1}\}$ and also $\{i_1, \ldots, i_{p-2}\} \subseteq \{j'_1, \ldots, j'_{p-1}\}$ and, by commutation, $a^{i_1 \cdots i_{p-2}}_{(j_1 \cdots j_{p-1})} = a^{i_1 \cdots i_{p-2}}_{(j'_1 \cdots j'_{p-1})}$. Thus to obtain the factorization γ: $D(n; p) \to R$ there is no ambiguity for terms of degree less than or equal to $p - 2$.

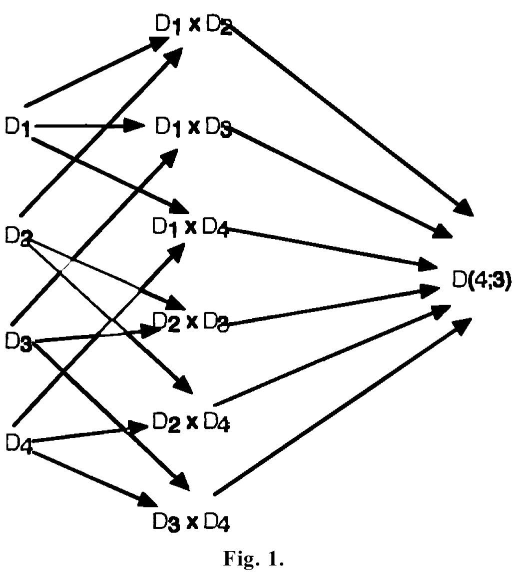

Fig. 1.

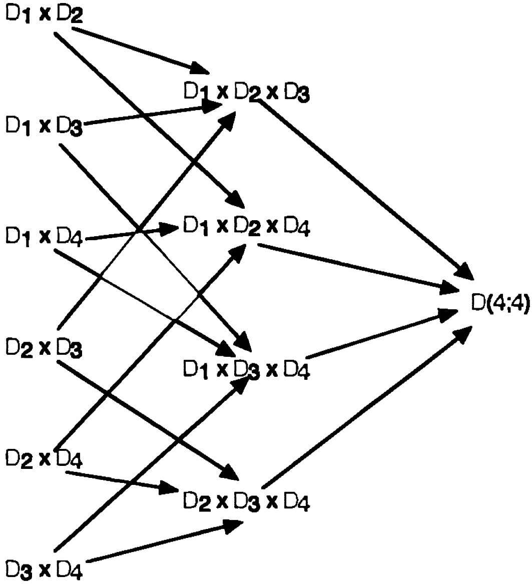

Fig. 2.

The terms of degree $p-1$ are the $b_{(j_1 \cdots j_{p-1})} d_{j_1} \cdots d_{j_{p-1}}$ of $\gamma_{j_1 \cdots j_{p-1}}$ for any $(j_1 \cdots j_{p-1})$ and there is no term of degree $>p-1$, as γ must only be defined on $D(n;p)$.

Unicity of γ is easy. ∎

We will now define the sum along an axis, the first, to fix one. Let γ_1 and γ_2 be two maps from $D(n;p)$ to M such that $\gamma_1(0, d_2, \ldots, d_n) = \gamma_2(0, d_2, \ldots, d_n)$. We consider the restrictions $\gamma_1^{j_1 \cdots j_{p-1}}$ and $\gamma_2^{j_1 \cdots j_{p-1}}$ of γ_1 and γ_2 to $D_{j_1} \times \cdots \times D_{j_{p-1}}$. We construct a new map $\gamma_1 +_1 \gamma_2$ from $D(n;p)$ to M by giving a cone on the diagram of which $D(n;p)$ is a quasicolimit. Let $1 \leq j_1 < j_2 < \cdots < j_{p-1} \leq n$. If $j_1 \neq 1$, we take $\gamma_{(j_1 \cdots j_{p-1})} = \gamma_1^{j_1 \cdots j_{p-1}} = \gamma_2^{j_1 \cdots j_{p-1}}$. If $j_1 = 1$, we take the sum along the first axis $\gamma_1^{j_1 \cdots j_{p-1}} +_1 \gamma_2^{j_1 \cdots j_{p-1}}$. Trivially we have a cone and the factorization $\gamma_1 +_1 \gamma_2$ is well defined.

If we consider the injection $D(n-1;p) \to D(n;p)$ inserting 0 at the kth position, the following is proved (for $k=1$, but it is true for any k);

Proposition 2. The mapping

$$M^{D(n;p)} \to M^{D(n-1;p)}$$

is a vector bundle. ∎

3. COMPARISON OF DIFFERENT CONCEPTS OF PUNCTUAL FORMS

For any small object of type $D(n; p)$ we have a punctual concept of a differential form.

Definition 1. Let M be a microlinear object. A differential $D(n; p)$-form is a map ω: $M^{D(n;p)} \to R$ which is homogeneous componentwise (and thus linear on each fiber) and which is alternated.

A $D(n; 2)$-form is what is called a classical differential n-form in . . . (19..) and a $D(n; n+1)$-form is called a singular differential n-form.

We denote by $\Omega_{D(n;p)}$ the set of differential $D(n; p)$-forms on M.

The chain of inclusions i_p

$$D(n) \to D(n; 3) \to \cdots \to D(n; p) \to D(n; p+1) \to \cdots \to D(n; n) \to D^n$$

induces, for any microlinear object M, the chain of retrictions M^{i_p}:

$$M^{D^n} \to M^{D(n;n)} \to \cdots \to M^{D(n;p+1)} \to M^{D(n;p)} \to \cdots \to M^{D(n;3)} \to M^{D(n)}$$

By composition with these restrictions, we obtain a map from $\Omega_{D(n;p)}(M)$ to $\Omega_{D(n;p+1)}(M)$ associating to ω its composite with restriction and this is, as ω, homogeneous componentwise and alternated (because the restrictions have properties of homogeneity and symmetry).

We ask for a map from $\Omega_{D(n;p+1)}(M)$ to $\Omega_{D(n;p)}(M)$. For that a section M^{i_p} would be useful.

Definition 2. An n-connection of degree p, ∇_n^p, is a section of M^{i_p},

$$\nabla_n^p\colon \quad M^{D(n;p)} \to M^{D(n;p-1)}$$

which is n-homogeneous (i.e., $\nabla_n^p(\alpha \cdot_k \gamma) = \alpha \cdot_k \nabla_n^p(\gamma)$ for any α in R, any γ in $M^{D(n;p)}$, and $1 \leq k \leq n$).

If we have ω in $\Omega_{D(n;p+1)}(M)$, we can consider $\omega \circ \nabla_n^p$, but this has the property of homogeneity, but not necessarily alternation. We ask for a condition of symmetry on ∇_n^p.

Let σ be a permutation of $(1, 2, \ldots, n)$. We define Σ: $M^{D(n;p)} \to M^{D(n;p)}$ by

$$\Sigma(\gamma)(d_1, \ldots, d_n) = \gamma(d_{\sigma(1)}, \ldots, d_{\sigma(n)})$$

Definition 3. We say an n-connection of degree p, ∇_n^p, is symmetric if

$$\nabla_n^p(\Sigma(\gamma)) = \Sigma(\nabla_n^p(\gamma))$$

for any γ in $M^{D(n;p)}$ and any permutation σ.

If M has a symmetric n-connection of degree p, ∇_n^p, we have the two maps

$$-\!\circ\, M^{i_p}\colon \quad \Omega_{D(n;p)}(M) \to \Omega_{D(n;p+1)}(M)$$

and

$$-\!\circ\, \nabla_n^p\colon \quad \Omega_{D(n;p+1)}(M) \to \Omega_{D(n;p)}(M)$$

We will show that they are inverse.

Some preliminaries are necessary. Let $X = D(n;\, p+1) \vee D((C_p^n)) = \{(\underline{d}, \underline{e}) \mid \underline{d} \in D(n;\, p+1),\ \underline{e} \in D((C_p^n)),\ \text{and}\ \forall_i \forall_j\ d_i \cdot d_j = 0\}$. We have two functions ϕ and ψ from $D(n;\, p+1)$ into X. They are given by:

(a) $\phi(\underline{d}) = (\underline{d}, 0)$
(b) $\psi(\underline{d}) = (\underline{d}, (d_{i_1} \cdot d_{i_2}, \ldots, d_{i_p})_{1 \le i_1 < \cdots < i_p \le n})$

Here we agree to order the sequences $1 \le i_1 < \cdots < i_p \le n$ by lexicographic order and to enumerate these sequences by integers from 1 to C_p^n. Then $\psi(\underline{d})$ is in X because $\underline{d} \in D(n;\, p+1)$ and thus the product of $p+1$ factors d_i from $\underline{d}$ is vanishing.

Proposition 3. The diagrams of Fig. 3 are quasicolimits.

Proof. Any $h{:}X \to R$ can be written uniquely as

$$h(\underline{d}, \underline{e}) = a_0 + \sum_{i=1}^{n} a_i d_i + \sum_{i<j} a_{ij} d_i d_j + \cdots$$

$$+ \sum_{1 \le i_1 < \cdots < i_p \le n} a_{i_1 \cdots i_p} d_{i_1} d_{i_2} \cdots d_{i_p} + \sum_{k=1}^{C_p^n} x_k e_k$$

If γ_1 and γ_2: $D(n;\, p+1) \to R$ coincide on $D(n;\, p)$, they can differ only by their terms of degree p. We can write

$$\gamma_1(\underline{d}) = a_0 + \sum_{i=1}^{n} a_i d_i + \cdots + \sum_{i_1 < \cdots < i_{p-1}} a_{i_1 \cdots i_{p-1}} d_{i_1} \cdots d_{i_{p-1}}$$

$$+ \sum_{i_1 < \cdots < i_{p-1}} a_{i_1 \cdots i_p} d_{i_1} \cdots d_{i_p}$$

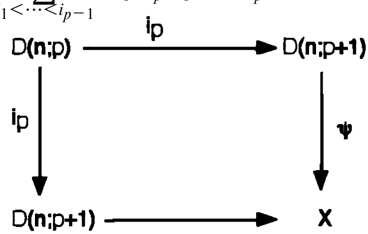

Fig. 3.

$$\gamma_2(\underline{d}) = a_0 + \sum_{i=1}^{n} a_i d_i + \cdots + \sum_{i_1<\cdots<i_{p-1}} a_{i_1\cdots i_{p-1}} d_{i_1} \cdots d_{i_{p-1}}$$
$$+ \sum_{i_1<\cdots<i_{p-1}} b_{i_1\cdots i_p} d_{i_1} \cdots d_{i_p}$$

Their common facorization through X is given by

$$g(\underline{d}, \underline{e}) = a_0 + \sum_{i=1}^{n} a_i d_i + \cdots$$
$$+ \sum_{i_1<\cdots<i_{p-1}} a_{i_1\cdots i_{p-1}} d_{i_1} \cdots d_{i_{p-1}} + \sum_{k=1}^{C_p^n} a_k p_k + \sum_{k=1}^{C_p^n} (b_k - a_k) e_k$$

where k is the number of the sequence $i_1 < \cdots < i_p$, $a_k = a_{i_1\cdots i_p}$, $b_k = b_{i_1\cdots i_p}$, and $p_k = d_{i_1} d_{i_2} \cdots d_{i_p}$. We have $g \circ \phi = \gamma_1$ and $g \circ \psi = \gamma_2$. It is easy to see that g is uniquely determined by these conditions.

The crucial step is then the following proposition.

Proposition 4. Let ω be a differential $D(n; p+1)$-form on M. If γ_1 and γ_2 [from $D(n; p+1)$ to M] coincide on $D(n; p)$, then $\omega(\gamma_1) = \omega(\gamma_2)$.

Proof. (a) By Proposition 3, as γ_1 and γ_2 coincide on $D(n; p)$, we have the factorization g: $X \to M$ with $g(\underline{d}, 0) = \gamma_1(\underline{d})$ and $g(\underline{d}, (d_{i_1} \cdots d_{i_p})_{i_1<\cdots<i_p}) = \gamma_2(\underline{d})$.

But we can compose g with other functions with values in X. We consider the $p+1$-microcube γ' given by

$$\gamma'(\underline{d} = g(\underline{d}, d_1 \cdot d_2 \ldots d_p, 0, 0, \ldots, 0)$$

(there are $C_p^n - 1$ zeros). For $d_1 = 0$, γ_1 and γ' coincide and we can consider $\gamma_1 -_1 \gamma'$ (the difference is taken along the first axis as described for Proposition 2).

(b) We shall now compute $\gamma_1 -_1 \gamma'$. We say that it is given by

$$(\gamma_1 -_1 \gamma')(\underline{d}) = g(0, d_2, \ldots, d_n, d_1 \cdot d_2 \ldots d_p, 0, \ldots, 0)$$

We consider the restrictions $\gamma_1^{1,2,\ldots,p}$ and $\gamma'^{,1,2,\ldots,p}$ of γ_1 and γ' to $D_1 \times \cdots \times D_p$ (where, as above, $D_i = D$ for all i). These restrictions are

$$\gamma_1^{1,2,\ldots,p}(d_1, d_2, \ldots, d_p, 0, \ldots, 0) = g(d_1, d_2, \ldots, d_p, 0, \ldots, 0)$$

(where the number of zeros is $n - p + C_p^n$); and

$$\gamma'^{,1,2,\ldots,p}(d_1, d_2, \ldots, d_p, 0, \ldots, 0)$$
$$= g(d_1, d_2, \ldots, d_p, 0, \ldots, 0, d_1 \cdot d_2 \cdots d_p, 0, \ldots, 0)$$

(where there are first $n - p$ zeros and then $C_p^n - 1$ zeros). We take now d_1 and d'_1 with $d_1 \cdot d'_1 = 0$. Consider $g(d_1 - d'_1, d_2, \ldots, d_p, 0, \ldots, 0, d_1 \cdot d_2 \cdots d_p, 0, \ldots, 0)$. For $d'_1 = 0$ we obtain $\gamma'^{,1,2,\ldots,p}$. For $d_1 = 0$ we obtain $g(-d'_1, d_2, \ldots, d_p, 0, \ldots, 0) = \gamma_1^{1,2,\ldots,p}(-d'_1, d_2, \ldots, d_p)$ and the difference of $\gamma_1^{1,2,\ldots,p}$ and $\gamma_1'^{,1,2,\ldots,p}$ is indeed $g(0, d_2, \ldots, d_p, 0, \ldots, d_1 \cdot d_2 \cdots d_p, 0, \ldots, 0)$.

(c) Let $\phi(\underline{d} = \gamma_1(0, d_2, \ldots, d_n)$, which is equal to $(\gamma_1 -_1 \gamma')(0, d_2, \ldots, d_n)$. For $d_2 = 0$ we have $\phi(d_1, 0, d_3, \ldots, d_n) = \gamma_1(0, 0, d_3, \ldots, d_n)$ and

$$\begin{aligned}(\gamma_1 -_1 \gamma')(d_1, 0, d_3, \ldots, d_n) &= g(0, 0, d_3, \ldots, d_n, 0, 0, \ldots, 0)\\ &= \gamma_1(0, 0, d_3, \ldots, d_n)\end{aligned}$$

and it makes sense to put

$$\alpha = (\gamma_1 -_1 \gamma') -_2 \phi$$

(d) We observe that $\alpha(d_1, 0, d_3, \ldots, d_n) = \alpha(0, d_2, d_3, \ldots, d_n) = \alpha(0, 0, d_3, \ldots, d_n)$ and thus α depends only on the product $d_1 \cdot d_2$ (Lavendhomme, 1996) Proposition 7 of Section 2.2). Thus α is symmetric in (d_1, d_2). As ω is alternated, $\omega(\alpha) = 0$. And, by the linearity along the second axis

$$0 = \omega((\gamma_1 -_1 \gamma') -_2 \phi)$$

and thus

$$\omega(\gamma_1 -_1 \gamma') = \omega(\phi)$$

But $\omega(\phi) = 0$ by the linearity of ω along the first axis (because ϕ is independent of d_1). By linearity on the first axis again we obtain

$$\omega(\gamma_1) = \omega(\gamma')$$

(e) We can repeat the process applied to γ' and γ_2 using now

$$\gamma^2(\underline{d}) = g(\underline{d}, 0, d_1 \cdot d_2 \cdots d_{p-1} \cdot d_{p+1}, 0, \ldots, 0)$$

and we obtain as above $\omega(\gamma') = \omega(\gamma^2)$.

We must apply this process C_p^n times (one time for each product $d_{i_1} \cdot d_{i_2} \cdots d_{i_p}$) and we construct $\gamma', \gamma^2, \gamma^3, \ldots, \gamma^{C_p^n}$. Observing that $\gamma^{C_p^n} = \gamma_2$, we obtain

$$\omega(\gamma_1) = \omega(\gamma_2)$$

as was asked. ■

Proposition 5. Let (M, ∇_n^p) be a microlinear object equipped with a symmetric n-connection of degree p. The maps

$$-\circ M^{i_p}: \quad \Omega_{D(n;p)}(M) \to \Omega_{D(n;p+1)}(M)$$

and

$$-\circ \nabla_n^p: \quad \Omega_{D(n;p+1)}(M) \to \Omega_{D(n;p)}(M)$$

are inverse bijections.

Proof. In one sense it is trivial and in the other it is a consequence of the preceding proposition. ■

Definition 4. An n-connection ∇_n on a microlinear object M is a family

$$(\nabla_n^p)_{2 \leq p \leq n}$$

where ∇_n^p is an n-connection of degree p. We say ∇_n is symmetric if each ∇_n^p is symmetric.

The following proposition is then a trivial corollary of Proposition 5.

Proposition 6. Let (M, ∇_n) be a microlinear object with a symmetric n-connection. There is a bijection between the sets of differential $D(n; p)$-forms for all p, in particular between classical and singular differential n-forms.

4 COMPARISON BETWEEN PUNCTUAL AND GLOBAL DIFFERENTIAL FORMS

We recall (Lavendhomme, 1996, Section 1.3, Definition 7) that a *diffusion* P on a microlinear object M is for each x in M an R-linear map

$$P_x: \quad T_xM \to X(M)$$

such that $P_x(t)(x) = t$.

A global differential n-form is a map ω: $\mathcal{X}(\mathcal{M})^{\backslash} \to \mathcal{R}^{\mathcal{M}}$ which is n-homogeneous (for products by elements of R^M) and alternated (Lavendhomme, 1996, Section 6.1, Definition 3).

Definition 5. Let M be a microlinear object and ω a global differential n-form. We say ω is *localizable* if $\omega(X_1, \ldots, X_n)(x) = 0$ if one of the vector fields $X_1, \ldots, X_n$ is singular in x (i.e., is vanishing at x). We denote by $\Omega_l^n(M)$ the set of localizable global differential n-forms.

Proposition 7. If (M, P) is a microlinear object M with a diffusion P, there exists a bijection between the set $\Omega^n(M)$ of (punctual) classical differential n-forms and the set $\Omega_l^n(M)$ of localizable global differential n-forms.

Proof. (a) We define $\phi: \Omega^n(M) \to \Omega_l^n(M)$ by

$$\phi(\omega)(X_1, \ldots, X_n)(x) = \omega_x(X_{1,x}, \ldots, X_{n,x})$$

Trivially $\phi(\omega)$ is a global differential n-form and it is localizable since $\omega_x(X_{1,x}, \ldots, X_{n,x})$ is vanishing if one of the $X_{i,x}$ is zero (by the R-n-homogeneity).

We define $\psi: \Omega_l^n(M) \to \Omega^n(M)$ by

$$\psi(\omega)_x(t_1, \ldots, t_n) = \omega(P_x t_1, \ldots, P_x t_n)(x)$$

for $t_1, \ldots, t_n$ in $T_x M$. And $\psi(\omega)$ is R-n-homgeneous and alternated.

(b) For ω in $\Omega^n(M)$,

$$\begin{aligned}\psi(\phi(\omega))_x(t_1, \ldots, t_n) &= \phi(\omega(P_x t_1, \ldots, P_x t_n)(x) \\ &= \omega_x((P_x t_1)_x, \ldots, (P_x t_n)_x) \\ &= \omega_x(t_1, \ldots, t_n)\end{aligned}$$

by definition of a diffusion. Thus $\psi \circ \phi = id_{\Omega^n(M)}$.

(c) For ω in $\Omega_l^n(M)$,

$$\begin{aligned}\phi(\psi(\omega))(X_1, \ldots, X_n)(x) &= \psi(\omega)_x(X_{1,x}, \ldots, X_{n,x}) \\ &= \omega(P_x(X_{1,x}, \ldots, P_x(X_{n,x}))(x)\end{aligned}$$

As $X_i - P_x(X_{i,x})$ vanish at x and ω is localizable, we have $\omega(X_1 - P_x(X_{1,x}), \ldots)(x) = 0$ and thus $\omega(P_x(X_{1,x}), \ldots, P_x(X_{n,x}))(x) = \omega(X_1, \ldots, X_n)(x)$ and so $\phi(\psi(\omega)) = \omega$. ■

We end with the banal observations that the exterior product of two localizable global differential forms is localizable, that the exterior differential of a localizable is localizable, and that the Lie derivative of a localizable is localizable.

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REFERENCES

Kock, A. (1978). *Differential Forms in Synthetic Differential Geometry*, Aarhus Preprint Series No. 28.

Kock, A. (1981). *Synthetic Differential Geometry*, Cambridge University Press, Cambridge.

Lavendhomme, R. (1987). *Leçons de Géometrie différentielle Synthétique Naïve*, Ciaco, Louvain-la-Neuve, France.

Lavendhomme, R. (1996). *Basic Concepts in Synthetic Differential Geometry*, Kluwer, Dordrecht.

Nishimura, H. (1997). *Theory of microcubes, International Journal of Theoretical Physics*, **36**, 1099–1131.